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ANALYSIS OF COMPLEX REDUNDANT STRUCTURES BY THE FLEXIBILITY METHOD

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ABSTRACT

This report presents the flexibility matrix method of structural analysis in detail. The method is extended to permit analysis of complex systems which consist of statically indeterminate component structures, connected to each other so that some of the connecting forces are redundant.

Digital program 64D017 implements the theory presented in this report and can be used for analysis of any complex structure.

SECTION I - INTRODUCTION

The structures of aerospace vehicles, of which the Saturn is a prime example, are often large and usually statically indeterminate. Those structures which can be idealized by an assemblage of one or two dimensional elements may be analysed by either the flexibility matrix method or the stiffness matrix method. Because of the large number of elements and the relatively small number of redundant forces in vehicles of the Saturn type, the flexibility matrix method is preferred. Also, the numerical problem of inverting large matrices is eliminated.

This report develops a method of obtaining the flexibility matrix of such structures by elastically coupling redundant component structures into a complex structure. Although this concept is not novel[1, 2], it is felt that the detailed ideas which make this method practical have not been sufficiently explained. Therefore, the method is rederived and particular attention is given to definitive statements regarding the nature and method of calculating the internal force influence matrices which are obtainable from equilibrium conditions, and which transform external unit applied loads or redundants into forces on the component structures or into internal forces on their structural elements. This includes axis transformation for elements in three dimensional space and an organized method that categorizes the various force influences, so that the force influence matrices that are the "coupling" matrices are easily understandable and calculable. This latter method is given in Section III, "Statically Indeterminate Coupling of Redundant Components of a Complex Structure." The practical application of the method makes the use of digital computers mandatory to perform the various matrix operations. The input data is in the form of the force influence matrices and flexibility matrices of standard structural elements. Novel idealizations are often possible which yield flexibility matrices that allow superior representations of the compatibility or equilibrium conditions where structure elements join. Therefore, the method is set up in such a way that any new element force and flexibility matrices can be used as they are calculated, without having to modify the basic digital program.

The redundant - internal force - and deflection influence coefficient matrices are derived, using the equality of internal and external work of deformation.

A procedure suggests itself which will permit the build up of the matrices of extremely complex structures from solutions of statically indeterminate structural subdivisions of reasonable complexity, requiring only the inversion of small matrices and various elementary matrix operations. The matrices involved are flexibility matrices of the simplest structural elements comprising the components, and internal force influence matrices.

The method is implemented by Digital Program No. 64 D 017. Instructions for its use will be given in a supplementary report. However, a sample problem is given in this report which illustrates the application of the theory.

SECTION II - FLEXIBILITY OF STRUCTURAL ELEMENTS

A. THE BEAM ELEMENT FLEXIBILITY MATRIX

The cantilevered beam element with two axes of cross-sectional symmetry is assumed to be the smallest basic element of that part of a structural network consisting of beams. The static deflection response of such an element to unit forces applied at its free end is expressed by its flexibility matrix, which is the matrix of coefficients $\left[\gamma_{\ell m}\right]$ of the generalized forces in the expression of the deflections:

$$\left[\gamma_{\ell m}\right]\left\{F_m\right\} = \left\{\Delta_m\right\} \tag{A. 1}$$

The calculation of the elements of $\left[\gamma_{\ell m}\right]$ is based both on the Principle of Virtual Work and the assumption that the internal stresses and strains are linearly dependent, on the basis of the engineering beam theory.

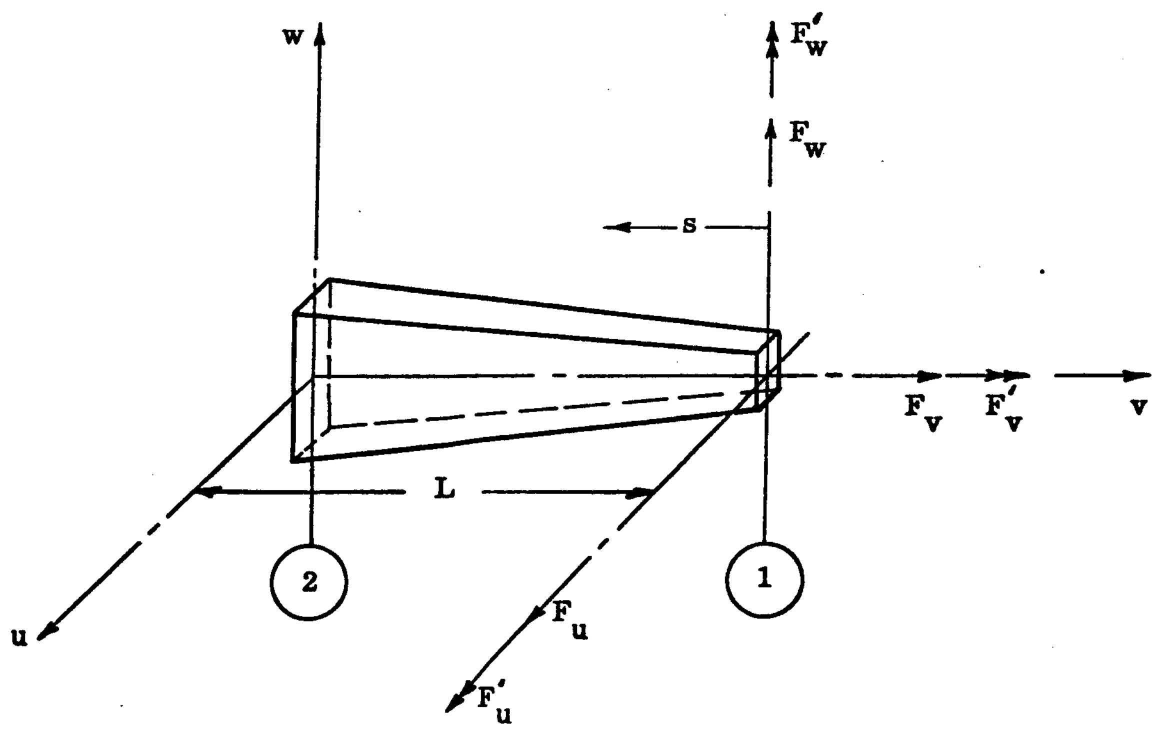


Figure A-1. Beam - Rod Element

Figure A-1 shows a beam element (1) - (2) fixed at its origin and free at point (1). The coordinate system coincides with the axes of symmetry. The six degrees of freedom of point (1) corresponds to the six components of $\{F_m\}$ shown in the figure. Thus

$$\{\Delta_m\} = \begin{Bmatrix} \Delta_u \\ \Delta_v \\ \Delta_w \\ \Delta'_u \\ \Delta'_v \\ \Delta'_w \end{Bmatrix}$$

and

$$\{F_m\} = \begin{Bmatrix} F_u \\ F_v \\ F_w \\ F'_u \\ F'_v \\ F'_w \end{Bmatrix}$$

where Δ = deflections

Δ' = rotations

F = forces

F' = moments

The cross-section properties at any point along the length of the beam are:

A_u = effective shear area loaded by F_u

A_v = effective axial area loaded by F_v

A_w = effective shear area loaded by F_w

I_u = moment of inertia about the u axis

I_v = torsional moment of inertia about the v axis

I_w = moment of inertia about the w axis

The non-zero elements of $\left[\gamma_{\ell m}\right]$ are:

$$\gamma_{uu} = \int_0^L \left(\frac{s^2}{EI_w} + \frac{1}{GA_u} \right) ds$$

$$\gamma_{vv} = \int_0^L \frac{ds}{EA_v}$$

$$\gamma_{ww} = \int_0^L \left(\frac{s^2}{EI_u} + \frac{1}{GA_w} \right) ds$$

$$\gamma_{u'w} = \gamma_{wu'} = \int_0^L \frac{sds}{EI_u}$$

$$\gamma_{u'u'} = \int_0^L \frac{ds}{EI_u}$$

$$\gamma_{v'v'} = \int_0^L \frac{ds}{GI_v}$$

$$\gamma_{w'u} = \gamma_{uw'} = -\int_0^L \frac{sds}{EI_w}$$

$$\gamma_{w'w'} = \int_0^L \frac{ds}{EI_w}$$

All other elements are zero.

Since practicality is of prime importance, it is recommended that the indicated integrations be performed by assuming that the elastic properties vary linearly between each pair of given numbers of point on each beam segment. The expanded form of Eq. (A. 1) is thus

$$
\begin{bmatrix}
\gamma_{uu} & & & \text{(Symmetric about diagonal)} & & \\
0 & \gamma_{vv} & & & & \\
0 & 0 & \gamma_{ww} & & & \\
0 & 0 & \gamma_{u'w} & \gamma_{u'u'} & & \\
0 & 0 & 0 & 0 & \gamma_{v'v'} & \\
\gamma_{w'u} & 0 & 0 & 0 & 0 & \gamma_{w'w'}
\end{bmatrix}
\begin{Bmatrix} F_u \\ F_v \\ F_w \\ F'_u \\ F'_v \\ F'_w \end{Bmatrix}
=
\begin{Bmatrix} \Delta_u \\ \Delta_v \\ \Delta_w \\ \Delta'_u \\ \Delta'_v \\ \Delta'_w \end{Bmatrix}
$$

B. THE ROD ELEMENT FLEXIBILITY MATRIX

A rod has only one degree of freedom, that of elongation of one end with respect to the other. If the beam element of Figure (A-1) is considered with only that degree of freedom, i.e., Δ_v , then the flexibility of the rod is γ_{vv}. Thus

$$\gamma_{vv} \; F_v = \Delta_v \tag{A.2}$$

where $\gamma_{vv} = \int_0^L \frac{ds}{EA_v}$

However, in the case of interaction of rods and shear panels, it is important to include the deflection of the rod due to unit value of an applied constant shear flow (Figure A-2).

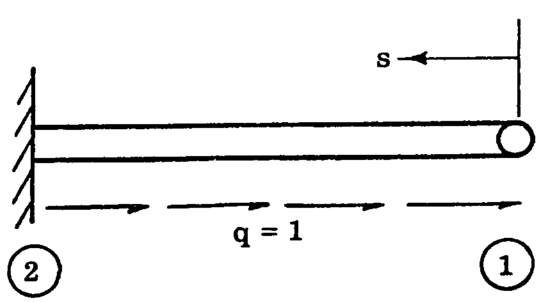


Figure A-2. Rod with Constant Shear Flow

The value of the deflection due to unit shear flow, where the subscript v″ applies to the shear flow, is

$$\gamma_{vv''} = \int_0^L \frac{s\,ds}{EA_v} \tag{A. 3}$$

Maxwell's Law of Reciprocity requires the existence of the term $\gamma_{v''v} = \gamma_{vv''}$. This is a generalized deformation due to the application of a unit end load, or otherwise interpreted, it is the work done by the unit shear flow as it displaces through deformations caused by the unit end load.

The corresponding diagonal term, $\gamma_{v''v''}$, is obtained by considering the applied load to be a unit shear flow and calculating the virtual work caused by that load.

Then

$$\gamma_{v''v''} = \int_0^L \frac{s^2 ds}{EA_v} \tag{A.4}$$

This is the work done by the unit shear flow as it moves through the deformation caused by it.

Thus

$$\left[\gamma\right]_{rod} = \begin{bmatrix} \gamma_{ii} & \gamma_{ii''} \\ \gamma_{i''i} & \gamma_{i''i''} \end{bmatrix} \tag{A.5}$$

where

$$i = u, v \text{ or } w$$

C. THE SHEAR PANEL FLEXIBILITY MATRIX

Some two dimensional elements in the form of skin panels or beam webs can be assumed to carry only pure constant shear flow (Figure A-3).

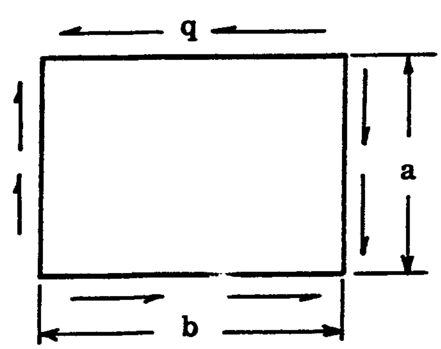


Figure A-3. Shear Panel

The shearing deformation for a unit shear flow is given by:

$$\gamma_{qq} = \frac{ab}{Gt} \qquad \text{(A. 6)}$$

If the panel has a rhomboid shape (Figure A-4), the skewed coordinates demand adjustment in this value according to the theory of elasticity.

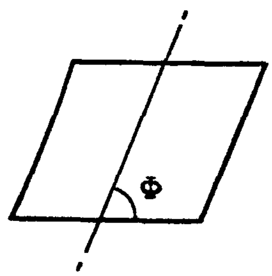


Figure A-4. Rhomboid Shear Panel

The flexibility is thereby increased and the value of the shearing deformation is then

$$\gamma_{qq} = \frac{ab}{Gt} \left(1 + \frac{G}{E} \cot^2 \Phi\right) \qquad \text{(A. 7)}$$

where
E = Young's modulus
G = shear modulus
Φ = angle of midline with respect to the side of the panel

The complementary strain energy is thus

$$U^* = \frac{1}{2} \gamma_{qq} q^2$$

D. FLEXIBILITY MATRICES OF ELEMENTS OF OTHER SHAPES

The flexibilities of rod and beam elements with various simple variations of cross section properties are given in Reference 3. The trapezoidal quadrilateral shear panel flexibilities are usually approximated by assuming equilibrium to be satisfied by suitably adjusted uniform shear flows on the edges.

Some of these approximations also appear in Reference 3. However, if a trapezoid is swept, similar to a rhomboid shape shear panel, the effect of this sweep must be incorporated by increasing the flexibilities, through the factor C, involving E, G and the average angle of sweep Φ:

$$C = \left(1 + 4 \frac{G}{E} \cot^2 \Phi\right)$$

SECTION III - STATICALLY INDETERMINATE COUPLING OF REDUNDANT COMPONENTS OF A COMPLEX STRUCTURE

In previous consideration of the analysis of statically indeterminate structures [3] internal forces were calculated which were caused by externally applied forces and redundants, respectively, on the cut (and thus statically determinate) structure. The concepts used are extended to the calculation of influence coefficients of complex structures consisting of statically indeterminate component structures, coupled so that redundant forces exist at the boundaries between the component structures.

A. ANALYSIS OF COMPONENT STRUCTURES

1. Basic Concepts for Solution of Component Structures

The structure is assumed to consist of interconnected elements in which the internal forces are statically determinate when sufficient cuts are made to remove redundancies (Figure A-5). Their individual idealizations permit the calculation of flexibility influence coefficients. These influence coefficients are used to calculate the deformations of the elements caused by applied loads. Linear structural behavior is assumed, permitting application of the superposition principle.

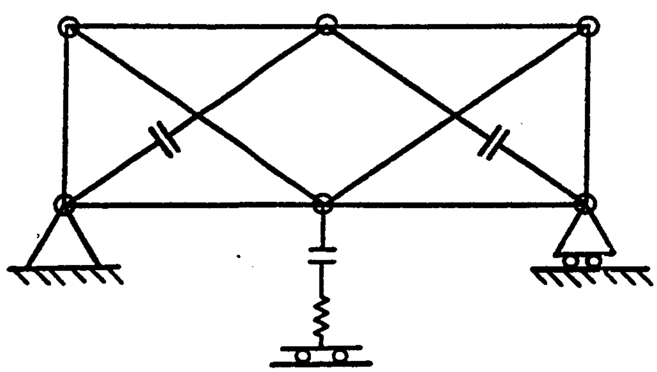


Figure A-5. Removal of Redundancies Through the Use of Cuts

Each element admits only a limited number of forces and corresponding kinematic motions. Examples are the slope and deflection of one end of a cantilever with respect to the fixed end caused by unit moments and shears. The deformation of the elements, with respect to their individual datums, are easily calculated by multiplying the matrices of the influence coefficients $[\gamma]$ of each of the elements by the force vector representing the forces $\{F\}$ sustained by it. These forces are due to external loads acting on the element or the internal forces due to load transfer between elements in the cut structure. The latter (e.g., shears and bending moments in a beam), can be determined of course from equations of static equilibrium of the structure that has been cut at all points of redundancy so that these internal forces are expressible in terms of the applied and redundant forces. The contributions to the internal forces from these two sources are expressed separately by the matrices $[\alpha_{mh}]$ and $[\beta_{mq}]$. The elements $[\alpha_{mh}]$ are the internal forces at m due to a unit load applied at h. Thus, when $[\alpha_{mh}]$ is multiplied by the external forces $\{P_h\}$, the resulting values are the internal forces $\{F_{mh}\}$ at interconnection points or loading points m. Thus,

$$\{F_{mh}\} = [\alpha_{mh}]\{P_h\} \tag{A.8}$$

Similarly, there are the internal forces $\{F_{mq}\}$ caused by the redundants $\{X_q\}$. The columns of the matrix $[\beta_{mq}]$ are the internal forces at m due to unit values of the redundants $\{X_q\}$ at q. Therefore,

$$\{F_{mq}\} = [\beta_{mq}]\{X_q\} \tag{A.9}$$

The total internal force is

$$\{F_m\} = \{F_{mh}\} + \{F_{mq}\} \tag{A.10}$$

or

$$\{F_m\} = \left[\alpha_{mh} \;\middle|\; \beta_{mq}\right] \left\{\frac{P_h}{X_q}\right\} \tag{A.11}$$

A propped cantilever beam, loaded at two points, is shown in Figure A-6. The beam is broken up into two elements and a cut is made between the beam and the flexible support. The internal shears v_1 and v_2, the moment m_2, and the support force p are the forces transformed through α_{mh} and β_{mq} shown below.

$$\{F_m\} = \begin{Bmatrix} v_1 \\ v_2 \\ m_2 \\ p_1 \end{Bmatrix} = \left[\alpha_{mh} \;\vdots\; \beta_{mq}\right] \begin{Bmatrix} P_h \\ \hline X_q \end{Bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 1 & 1 & -1 \\ \ell_1 & 0 & -\ell_1 \\ 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} P_1 \\ P_2 \\ \hline X_1 \end{Bmatrix}$$

Thus

$$\left[\alpha_{mh}\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \ell_1 & 0 \\ 0 & 0 \end{bmatrix} ; \left[\beta_{mq}\right] = \begin{bmatrix} -1 \\ -1 \\ -\ell_1 \\ 1 \end{bmatrix}$$

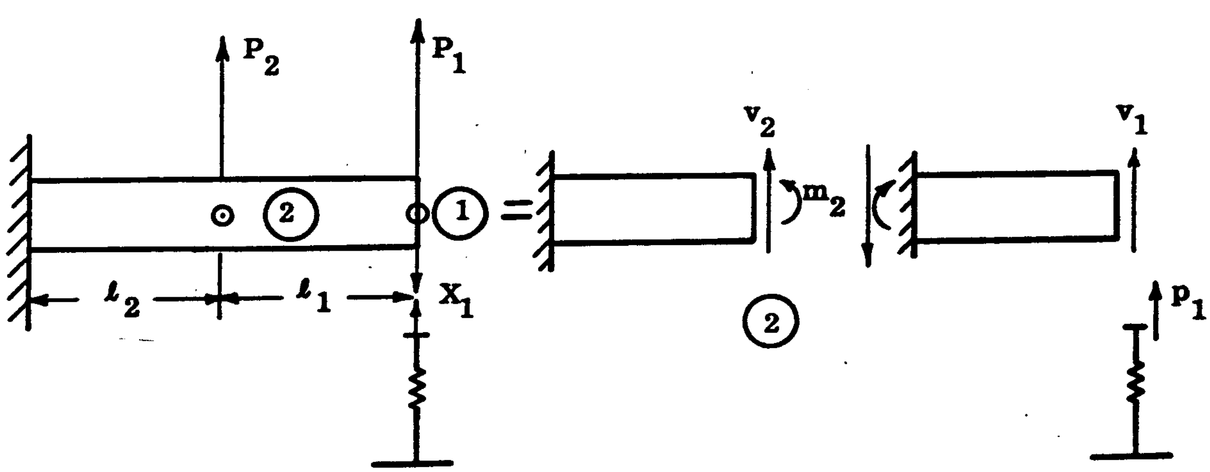


Figure A-6. Internal Forces in a Redundant Beam

Methods for finding $\{X_q\}$, and thus $\{F_m\}$, will now be obtained by considering the equality of the internal and external work of the structural deformations.

2. Internal Work

The internal work of the structure is obtained by summing the contributions of each element to that work. Only the work of the internal and external forces in and on an element relative to the element datum will be used to find the internal work. Actually, the absolute displacements, which are the sum of the datum and the relative displacements, could be used to find the work but the contribution of rigid body motion is zero because the complete force system on each element, including the reactions at its datum, is in equilibrium and the work required for a rigid body displacement of a system of forces in equilibrium is equal to zero.

The deflections of an element relative to its datum are

$$\left\{\Delta_\ell\right\} = \left[\gamma_{\ell m}\right] \left\{F_m\right\} \tag{A. 12}$$

It is to be understood that, physically, ℓ and m may represent the same or different degrees of freedom; ℓ can be a deflection or a rotation type deformation and m can be a linear or a moment type force.

The structure is assumed to behave linearly and therefore the internal work W_{el}, the product of these forces and the deflections, is given by

$$W_{el} = \frac{1}{2} \left\{F_m\right\}^T \left\{\Delta_\ell\right\} \tag{A. 13}$$

Substituting Eqs. (A. 11) and (A. 12) into (A. 13)

$$W_{el} = \frac{1}{2} \left[\left[P_h\right]^T \middle| \left[X_q\right]^T\right] \begin{bmatrix} [\alpha_{mh}]^T \\ \hline [\beta_{mq}]^T \end{bmatrix}_{(el)} \left[\gamma_{\ell m}\right]_{(el)} \left[[\alpha_{mh}] \middle| [\beta_{mq}] \right]_{(el)} \begin{Bmatrix} P_h \\ \hline X_q \end{Bmatrix} \tag{A. 14}$$

The total internal work in the structure is the sum of the work contributions of the elements:

$$W_{int} = \sum_{j=1}^{n} W_{el_j} \tag{A. 15}$$

where

n = number of elements

Now $\left\{\frac{P_h}{X_q}\right\}$ will be common to all expressions of internal work for all structural elements such as Eq. (A.14), provided $[\alpha_{mh}]$ and $[\beta_{mq}]$ are organized to have columns corresponding to every P_h and X_q sustained by the structure, even if some of these forces have zero influence. The sum of the internal works, Eq. (A.15), is obtained by providing enough rows in $\left[[\alpha_{mh}] \mid [\beta_{mq}]\right]$ and $[\gamma_{\ell m}]$ so that all internal forces in the component structure and their flexibilities are represented.

The deflections of a structure consisting of two elements would be:

$$\{\Delta_\ell\}_i = \left[\gamma_{\ell m}\right]_i \left[[\alpha_{mh}] \mid [\beta_{mq}]\right]_i \left\{\frac{P_h}{X_q}\right\}$$

$$\{\Delta_\ell\}_{i+1} = \left[\gamma_{\ell m}\right]_{i+1} \left[[\alpha_{mh}] \mid [\beta_{mq}]\right]_{i+1} \left\{\frac{P_h}{X_q}\right\}$$

or

$$\left\{\begin{array}{c}\{\Delta_\ell\}_i \\ \hline \{\Delta_\ell\}_{i+1}\end{array}\right\} = \left[\begin{array}{c|c}\left[\gamma_{\ell m}\right]_i & [0] \\ \hline [0] & \left[\gamma_{\ell m}\right]_{i+1}\end{array}\right] \left[\begin{array}{c|c}\left[\alpha_{mh}\right]_i & \left[\beta_{mq}\right]_i \\ \hline \left[\alpha_{mh}\right]_{i+1} & \left[\beta_{mq}\right]_{i+1}\end{array}\right] \left\{\begin{array}{c}P_h \\ \hline X_q\end{array}\right\} \tag{A.16}$$

Note that the elements in the i^{th} set of rows of $[\alpha_{mh}]$ must therefore be the values of the internal forces in the i^{th} element due to unit values of external loads $\{P_h\}$ applied anywhere on the cut, statically determinate structure. The elements in the i^{th} set of rows of $[\beta_{mq}]$ are similarly the internal forces in the i^{th} element due to unit values of the redundants $\{X_q\}$.

The internal work is one half the product of the internal forces and the deflections. This work in two elements is:

$$W_{int} = W_{el_i} + W_{el_{i+1}}$$

$$= \frac{1}{2}\left[F_{m_i}^T \Delta_{m_i} + F_{m_{i+1}}^T \Delta_{m_{i+1}}\right] \tag{A.17}$$

$$= \frac{1}{2}\left[\{F_{m_i}\}^T \mid \{F_{m_{i+1}}\}^T\right] \left\{\begin{array}{c}\Delta_{m_i} \\ \hline \Delta_{m_{i+1}}\end{array}\right\}$$

But

$$\left[\left\{F_{m_i}\right\}^T \,\middle|\, \left\{F_{m_{i+1}}\right\}^T\right] = \left[\left\{P_h\right\}^T \,\middle|\, \left\{X_q\right\}^T\right] \left[\begin{array}{c|c} [\alpha_{mh}]_i^T & [\alpha_{mh}]_{i+1}^T \\ \hline [\beta_{mq}]_i^T & [\beta_{mq}]_{i+1}^T \end{array}\right] \qquad (A.18)$$

Thus, for two elements,

$$W_{int} = \frac{1}{2}\left[P_h^T \,\middle|\, X_q^T\right] \left[\begin{array}{c|c} \alpha_{mh_1}^T & \alpha_{mh_2}^T \\ \hline \beta_{mq_1}^T & \beta_{mq_2}^T \end{array}\right] \left[\begin{array}{c|c} \gamma_{\ell m_1} & 0 \\ \hline 0 & \gamma_{\ell m_2} \end{array}\right] \left[\begin{array}{c|c} \alpha_{mh_1} & \beta_{mq_1} \\ \hline \alpha_{mh_2} & \beta_{mq_2} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

(A.19)

in which various brackets and braces of the notation in (A.18) have been deleted for clarity. Each of the letter symbols now stands for the corresponding matrix.

Generalizing this to any number of elements, the total internal work is:

$$W_{int} = \frac{1}{2}\left[P_h^T \,\middle|\, X_q^T\right] \left[\begin{array}{c} \alpha_{mh}^T \\ \hline \beta_{mq}^T \end{array}\right] \left[\diagdown \, \gamma_{\ell m} \, \diagdown\right] \left[\alpha_{mh} \,\middle|\, \beta_{mq}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \qquad (A.20)$$

which has the same form as Eq. (A.14), but gives the generalized expression of internal work for the whole structure through the foregoing definitions of $[\alpha_{mh}]$ and $[\beta_{mq}]$ and by designating $[\gamma_{\ell m}]$ to be a square, symmetrical matrix whose elements are uncoupled flexibility matrices of the elements along the diagonal, such as that in Eq. (A.19) above.

3. <u>External Work</u>

The external work is expressed by summing the work of the external forces as they move through their displacements. It will be helpful to use the trick of "adding zero" in the derivation of the expression of the equality of internal and external work, so that terms involving the redundants in the accompanying matric equations may be understood.

The concept of an external force can be generalized to include the redundants which are applied to each side of a "cut" face, equal and opposite to each other. Compatibility of the cut faces requires that each side of the cut face move through the same absolute displacement. The external work of the redundants is therefore equal to zero because they are each equal and opposite forces, representing a zero vector, moving through some displacement with respect to an absolute datum. Addition of the "external work" of the redundants to the total external work is therefore adding zero.

The cut points and others which are loaded with external known forces also move because of the influence of the loads on them and the forcing of compatibility by the redundants. Then if the structure is cut so that all forces within it are statically determinate with respect to externally applied forces and the redundants, the resulting deflections are expressed through means of a flexibility matrix [c] referred to some common absolute datum for the structure. It is desired to obtain the displacements at the externally loaded points with respect to this datum. They are calculated from an equation, such as

$$\left\{\Delta_g\right\} = \left[c_{gh} \;\middle|\; c_{gq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A.21}$$

in which $\{\Delta_g\}$ are the deflections of the externally loaded points, g, and [c] is the flexibility matrix of the cut, statically determinate structure. The subscripts g and h pertain to applied loads or corresponding degrees of freedom at their points of application, and p and q pertain to redundants or the corresponding degrees of freedom at their points of application. Thus the partitions c_{gh} and c_{gq} of [c] are the g deformations due to unit values of forces P_h and X_q, respectively.

Consider the displacements $\left\{\Delta_p\right\}_{(a)}$ of one of the two faces of each cut. They can be similarly expressed as:

$$\left\{\Delta_p\right\}_{(a)} = \left[c'_{ph} \;\middle|\; c'_{pq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{a}$$

for the faces (a). For the faces (b) they would be:

$$\left\{\Delta_p\right\}_{(b)} = \left[c''_{ph} \;\middle|\; c''_{pq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{b}$$

The relative deformations of (a) and (b) are the differences. Thus

$$\left\{\Delta_p\right\} = \left\{\Delta_p\right\}_{(a)} - \left\{\Delta_p\right\}_{(b)}$$

The relative displacements of the cut faces can be expressed, as was done for the load point displacements, as

$$\left\{\Delta_p\right\} = \left[c_{ph} \,\middle|\, c_{pq}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A. 22}$$

in which

$$\left[c_{ph}\right] = \left[c'_{ph}\right] - \left[c''_{ph}\right]$$

and

$$\left[c_{pq}\right] = \left[c'_{pq}\right] - \left[c''_{pq}\right]$$

which shows the use of the primed and double primed matrices of Eqs. (a) and (b).

The total work of the forces is given by the sum of the products of the forces and their displacements. Remembering that $\{\Delta_p\}$ is defined as the relative displacement vector of the cut faces, then the total external work (if $\{\Delta_p\}$ were not equal to zero) is

$$W_{ext} = \frac{1}{2}\left[\left\{P_h\right\}^T \left\{\Delta_g\right\} + \left\{X_q\right\}^T \left\{\Delta_p\right\}\right]$$

or

$$W_{ext} = \frac{1}{2}\left[\left\{P_h\right\}^T \,\middle|\, \left\{X_q\right\}^T\right] \left\{\begin{array}{c} \Delta_g \\ \hline \Delta_p \end{array}\right\} \tag{A. 23}$$

Substituting Eq. (A. 21) and (A. 22) in Eq. (A. 23) gives

$$W_{ext} = \frac{1}{2}\left[P_h^T \,\middle|\, X_q^T\right] \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A. 24}$$

Now, recalling the expression of internal work, Eq. (A.20), and setting

$$W_{ext} = W_{int}$$

there results, as shown in Appendix A:

$$\left[\begin{array}{c} \alpha_{mh}^{T} \\ \hline \beta_{mq}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\alpha_{mh} \,\vdots\, \beta_{mq}\right] = \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] \qquad \text{(A.25)}$$

A means of calculating $[\,c\,]$ has thus been found. The values of the redundants and the deformations of the compatible structure remain to be found. The internal forces can be obtained from Eq. (A.11) once the redundants are known.

It can be seen from Eq. (A.21) that if all the values of $\{X_q\}$ for every separate application of a unit external load at point h were known, then the values of $\{\Delta_g\}$ for any such unit load would form, by definition, one column of the external flexibility influence coefficient matrix, i.e., that the resulting deflections are caused by a unit force applied to the structure at h, and in which compatibility is enforced by corresponding values of $\{X_q\}$.

To find these values of $\{X_q\}$, the relative deformations $\{\Delta_p\}$ in Eq. (A.22) are set equal to zero, i.e., for compatibility of the cuts:

$$\left\{\Delta_p\right\} = 0$$

Then

$$\left[c_{ph}\right]\left\{P_h\right\} + \left[c_{pq}\right]\left\{X_q\right\} = 0$$

Therefore

$$\left\{X_q\right\} = -\left[c_{pq}\right]^{-1}\left[c_{ph}\right]\left\{P_h\right\} \qquad \text{(A.26)}$$

Substituting this result in Eq. (A. 21), gives the deflections of points (g) as:

$$\left\{\Delta_g\right\} = \left[c_{gh} \;\middle|\; c_{gq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A. 21}$$

$$= \left[c_{gh} \;\middle|\; c_{gq}\right]\left\{\begin{array}{c} P_h \\ \hline -\,c_{pq}^{-1}\; c_{ph}\, P_h \end{array}\right\}$$

Thus

$$\left\{\Delta_g\right\} = \left[c_{gh} - c_{gq}\, c_{pq}^{-1}\; c_{ph}\right]\;\left\{P_h\right\} \tag{A. 27}$$

Let

$$\left[\gamma_{gh}\right] = \left[c_{gh} - c_{gq}\, c_{pq}^{-1}\; c_{ph}\right] \tag{A. 28}$$

Then

$$\left\{\Delta_g\right\} = \left[\gamma_{gh}\right]\left\{P_h\right\} \tag{A. 27a}$$

which shows that $[\gamma_{gh}]$, according to definition is the flexibility influence coefficient matrix of the redundant structure with respect to its own datum.

The internal forces $\{F_m\}$ can be calculated for unit values of the applied forces $\{P_h\}$. The result is the internal force influence matrix $[f_{mh}]$. Substituting $\{X_q\}$ of Eq. (A. 26) into Eq. (A. 11),

$$\left\{F_m\right\} = \left[\alpha_{mh} \;\middle|\; \beta_{mq}\right]\left\{\begin{array}{c} P_h \\ \hline -c_{pq}^{-1}\; c_{ph}\, P_h \end{array}\right\}$$

Thus

$$\left\{F_m\right\} = \left[\alpha_{mh} - \beta_{mq}\; c_{pq}^{-1}\; c_{ph}\right]\;\left\{P_h\right\} \tag{A. 29}$$

Let

$$\left[f_{mh}\right] = \left[\alpha_{mh} - \beta_{mq}\; c_{pq}^{-1}\; c_{ph}\right]$$

Then $$\{F_m\} = [f_{mh}] \{P_h\} \tag{A.29a}$$

which defines $[f_{mh}]$ as the internal force influence matrix of the uncut redundant structure.

4. Summary

Due to loads at points (h):

The redundants are

$$\{X_q\} = -[c_{pq}]^{-1} [c_{ph}] \{P_h\} \tag{A.30}$$

The internal forces are

$$\{F_m\} = [\alpha_{mh} - \beta_{mq}\, c_{pq}^{-1}\, c_{ph}] \{P_h\} \tag{A.31}$$

The deflections are

$$\{\Delta_g\} = [c_{gh} - c_{gq}\, c_{pq}^{-1}\, c_{ph}] \{P_h\} \tag{A.32}$$

It should be kept in mind that for certain structures consisting of many elements joining in few points it may be advantageous to obtain $[\gamma_{gh}]$ by inverting the stiffness matrix of the structure, which in such cases may be more easily obtainable (References 4 and 5).

B. ANALYSIS OF A STRUCTURAL COMPLEX

The method of solving statically indeterminate structures has been summarized previously in Eqs. (A.30) through (A.32). The method can now be extended to cover a class of structures that consist of statically indeterminate modules interconnected in a statically indeterminate way. It is proposed that through such a concept any extremely large structural complex can be solved by analyzing the properties of component structures, chosen at convenience, which may themselves be statically indeterminate, and coupling them through the satisfaction of equilibrium and compatibility conditions at their points of physical interconnection. The generalized relationships which lead to the expression of the interconnecting matrices will be developed, and as will be seen, the concepts that were used to develop Eqs. (A.30) through (A.32) are also applicable here. Furthermore, although this may be an extreme generalization, it can be seen that the properties of the statically indeterminate component structures may themselves have been obtained from a further breakdown into statically indeterminate sub-modules and a subsequent synthesis of the component structure through the compatible interconnection of these sub-modules. This can be continued ad infinitum, conceptually producing the picture-within-a-picture-within-a-picture effect.

The method will be developed assuming statically indeterminate component structures, interconnected in a statically indeterminate manner. It will subsequently be shown what the specialized terms of the various matrices are for structures that have component structures and interconnections that are statically determinate.

1. Forces on the Component Structure

The above title suggests that the forces on component structures might have different sources. This is indeed true. The correct generation of the corresponding force influence matrices is dependent upon the proper classification of the force sources and their effects. They are so-called "extractor" matrices,[6] because they extract the local force from the generalized loading matrix. These matrices are generated from the equations of static equilibrium, somewhat similarly to the calculation of α_{mh} and β_{mq} for the elements of the component structure. It is the use of these force influence matrices that distinguishes this approach from that of Argyris, [2] in which the internal element forces of the connected component structure are used to obtain the work in the components of the complex structure. The use of the new statically obtainable force influence matrices is obviously much simpler to understand as well as to actually implement.

A datum is chosen for each component structure so that the reactions there are statically determinate. At these points, each component structure is joined to another one. All other points on the component represent points where arbitrary loads can be directly introduced. Quantities at these points are designated by the subscript g_i, where g designates the degree of freedom at a point, e.g., a rotation or deflection, and i stands for the interior nature of the point, it being within the boundary of the component or on the boundary, but not connected to other structures there. The forces which correspond to the senses of these degrees of freedom are subscripted with h_i. For example, the numerical designation in an actual analysis of the deformation g_i, which may be a rotation, would be the same as that of the moment h_i if it were applied in the same location.

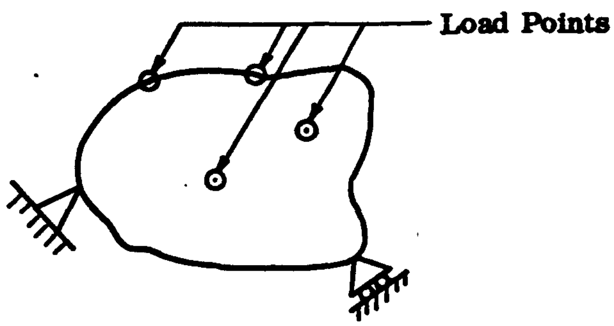


Figure A-7. Typical Load Points g_i, h_i on Component Structure

The points which represent the statically determinate connection of other component structures (i.e., their datums) to the presently considered structure are called g_b. At these points the negative of the reactions of the other components to external loads applied at any point k are introduced (Figure A-8). The subscript b stands for boundary between component structures. At such points, external forces P_{h_b} may be applied with the same sense as the corresponding deformational degrees of freedom there.

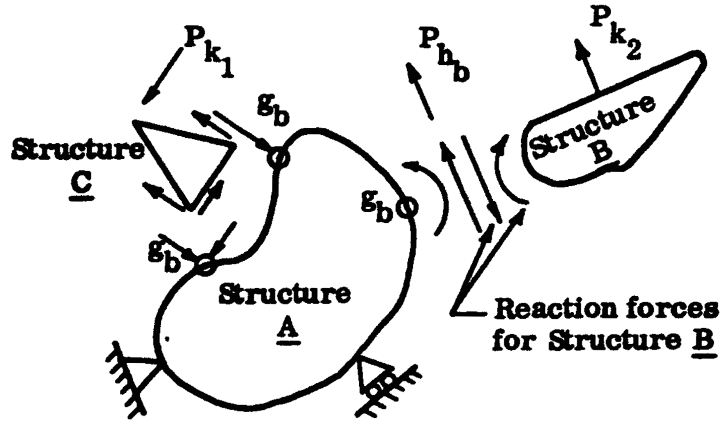


Figure A-8. Typical Load Points g_b

The third type of point is that at which the statically indeterminate coupling forces are introduced. These are called g_t (Figure A-9). These points may be externally loaded, which loads are designated P_{h_t}.

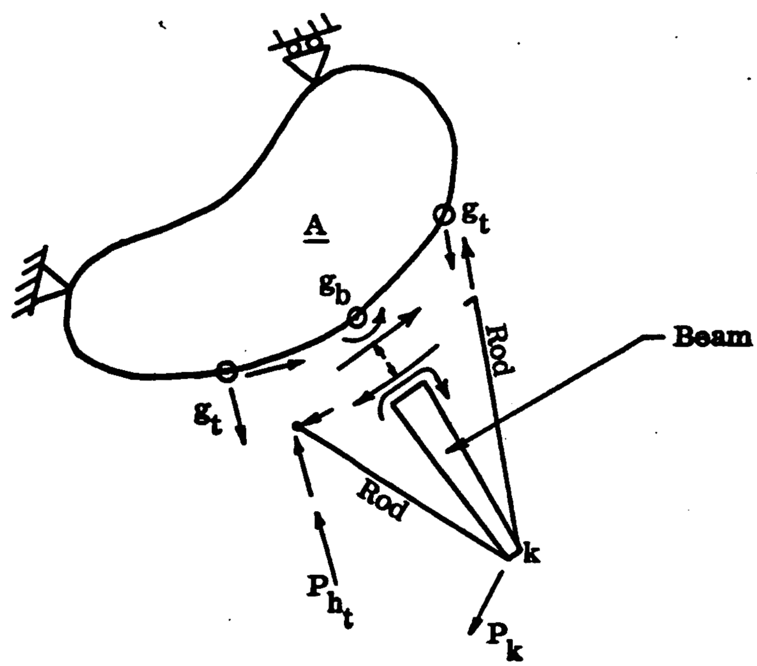


Figure A-9. Redundant Connection of Two Structures

In summary, the point nomenclature is as follows:

$g_{i_{(c)}}$ = Pertains to deformation degree of freedom g at a point within or on boundary of the structure designated (c).

$g_{b_{(c)}}$ = Pertains to deformation degree of freedom at points of statically determinate connection of (c) with other component structures, datum of (c) excluded.

$g_{t_{(c)}}$ = Pertains to deformation degrees of freedom at points of connecting redundant application on structure (c).

Similarily, the corresponding force nomenclature substitutes h for g in the subscripts, other items being held equal so that $h_{i_{(c)}}$, $h_{b_{(c)}}$, $h_{t_{(c)}}$ pertain to forces corresponding in sense and location to their deformation counterpart at $g_{i_{(c)}}$, $g_{b_{(c)}}$ and $g_{t_{(c)}}$, respectively.

It should be noted that the symmetry of the influence coefficient matrix $\left[\gamma_{gh}\right]$ (Eq. A.28) actually shows that g and h can be used interchangeably.

Having dealt with the force and deformation designations for a component structure, it will now be important to find a nomenclature for the general degrees of freedom and corresponding forces applied anywhere on a complex assembly of component structures. The reason is that we want to describe the response of the complex structure to applied forces in the same way as the response of the component structure was described. This is to say, it is desirable to talk about the degrees of freedom of points that are externally loaded and the degrees of freedom of redundant interconnections separately. Therefore, the following definitions are given

a. <u>Deformation Designations of the Complex Structure</u>

The following subscripts describe the nature of subscripted quantities:

n = Pertains to generalized deformation degrees of freedom anywhere on the complex structure, externally loaded or at which flexibility influence, coefficients are required. They include those designated h_i and h_b on the component structures. Excluded are those which correspond to redundants existing at the boundaries between component structures.

s = Pertains to deformation degrees of freedom of the points at the redundant cuts existing between component structures. These degrees of freedom have a subclass, designated by s_p, which are those of the cut points externally loaded by forces of the same sense and location as the corresponding degree of freedom, or at which flexibility influence coefficients are desired.

b. Loads on the Complex Structure

k, t = Pertains to forces corresponding in sense and location to their deformation counterparts n and s, respectively. This means that the forces designated by k are external forces and those designated by t are the redundants. The forces designated by t_p, corresponding to s_p, are externally applied at the location of interconnecting redundants with the sense of the corresponding degree of freedom. Their positive direction is the same as that for all other externally applied loads which have the same sense.

Using these definitions, a transformation matrix can be constructed which will express the forces applied to a component structure in terms of the forces applied to the complex structure.

The loads P_{h_i} on the structure of Figure A-2 due to forces at the general points are expressed through the transformation matrix $\left[a_{h_i k} \right]$ as follows:

$$\left\{ P_{h_i} \right\} = \left[a_{h_i k} \right] \left\{ P_k \right\}$$

where h_i are points on the component structure and k are all the points on the complex structure where forces are applied, or for which influence coefficients are to be computed.

c. <u>Illustrative Example of</u> $\left[a_{h_i k}\right]$

The loads on the boundary, directly applied, for the structure I of Figure A-10 are described by the matrix $\left[a_{h_i k}\right]$ in terms of the loads $\left\{P_k\right\}$, which is the complete load matrix for the complex structure.

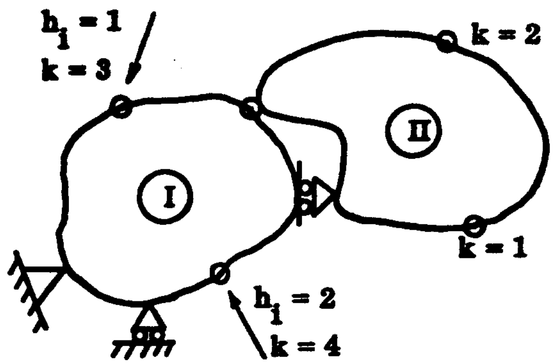


Figure A-10. Structure Loaded by Forces at h_i

$$\left[a_{h_i k}\right]_I = \begin{array}{c|cccc} h_i \backslash k & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \quad ; \quad \left\{P_k\right\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

so that $\quad \left\{P_{h_i}\right\}_I = \left[a_{h_i k}\right]_I \left\{P_k\right\}$

Thus

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}_I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

For the structure II of Figure A-10:

$$[a_{h_i k}] = \begin{array}{c|cccc} h_i \backslash k & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{array}$$

Thus

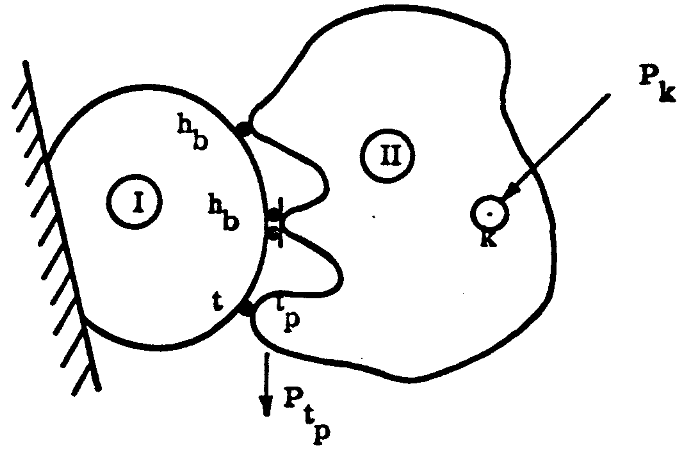
$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}_{II} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

Collectively we can thus write

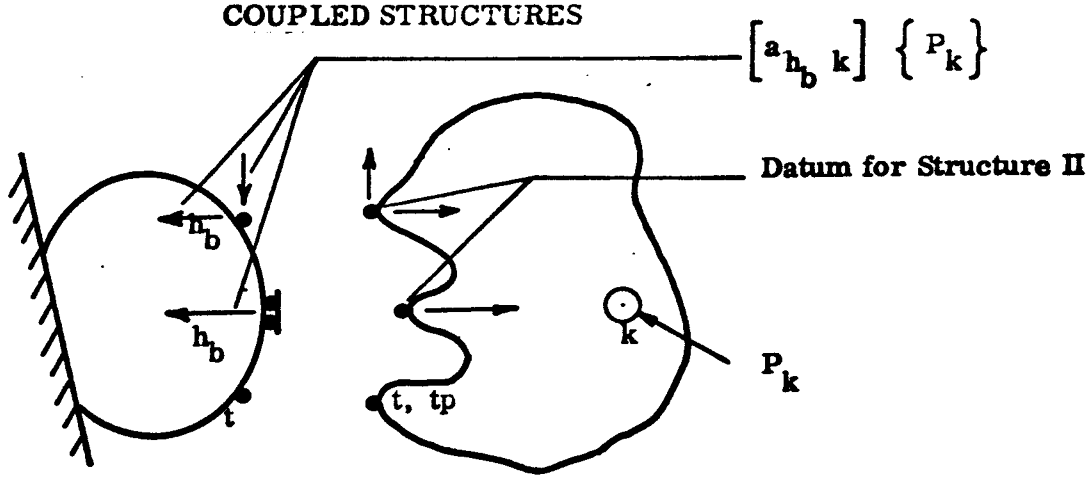
$$\begin{Bmatrix} \{P_{h_i}\}_I \\ \hline \{P_{h_i}\}_{II} \end{Bmatrix} = \begin{bmatrix} [a_{h_i k}]_I \\ \hline [a_{h_i k}]_{II} \end{bmatrix} \{P_k\}$$

The forces P_{h_b} have three possible sources as illustrated in Figure A-11. First, there are the reaction forces at h_b due to unit values of forces anywhere at k which form the matrix $[a_{h_b k}]$. Next, there are reaction forces at h_b due to unit values of forces of type t_p. These are given by $[a_{h_b t_p}]$. The third contribution to P_{h_b} is the set of reaction forces at h_b due to unit values of redundants at t. The matrix expressing these relations is $[b_{h_b t}]$. The loads on the structures from these three sources are the sum of their influences:

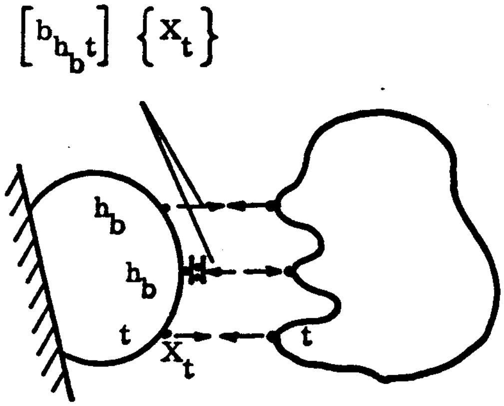
$$\{P_{h_b}\} = \left[[a_{h_b k}] \,\Big|\, [a_{h_b t_p}] \,\Big|\, [b_{h_b t}] \right] \begin{Bmatrix} P_k \\ \hline P_{t_p} \\ \hline X_t \end{Bmatrix}$$



COUPLED STRUCTURES

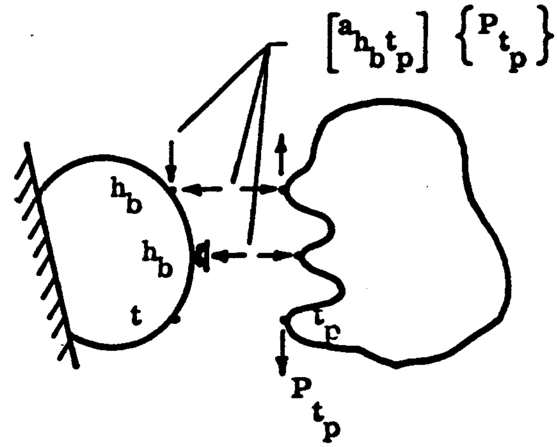


FORCES AT h_b DUE TO LOADS P_k
ON THE COUPLED STRUCTURE



FORCES AT h_b DUE TO

REDUNDANTS X_t



FORCES AT h_b DUE TO

LOADS P_{t_p}

A-11. Loads on h_b Points of Coupled Component Structures

Finally there are forces at h_t due to forces P_{h_t} applied on <u>one side</u> of the cut face that belongs to the presently considered component structure, and those caused by unit values of the redundants between component structures. The first are given by $\left[a_{h_t t_p}\right]\left\{P_{t_p}\right\}$ and the second by $\left[b_{h_t t}\right]\left\{X_t\right\}$, as shown in Figure A-12.

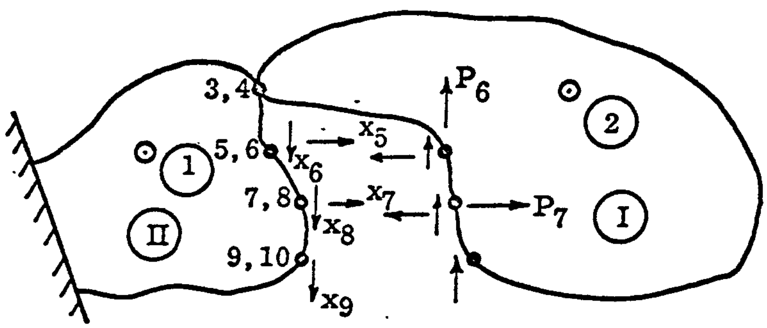


Figure A-12. Forces at h_t Due to Applied Loads and Redundants

$$
\left[a_{h_t t_p}\right]_I = \begin{array}{c|c} h_t \backslash t_p & \begin{array}{cc} 6 & 7 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}
\qquad
\left[b_{h_t t}\right]_I = \begin{array}{c|c} h_t \backslash t & \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}
$$

$$
\left[a_{h_t t_p}\right]_{II} = \begin{array}{c|c} h_t \backslash t_p & \begin{array}{cc} 6 & 7 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}
\qquad
\left[b_{h_t t}\right]_{II} = \begin{array}{c|c} h_t \backslash t & \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}
$$

Thus

$$\left\{P_{h_t}\right\}_{\mathrm{I}} = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{Bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & [\,I\,] \end{array}\right] \begin{Bmatrix} P_6 \\ P_7 \\ \hline X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{Bmatrix}$$

$$\left\{P_{h_t}\right\}_{\mathrm{II}} = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{Bmatrix} = \left[\begin{array}{c|c} [\,0\,] & -[\,I\,] \end{array}\right] \begin{Bmatrix} P_6 \\ P_7 \\ \hline X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{Bmatrix}$$

In summarizing these descriptions, the loads located at various points of the complex structure, cut so that component structures are interconnected in a statically determinate manner, cause forces on any particular component structure (c) which can now be expressed in the matric equation

$$\left\{P_h\right\}_{(c)} = \left\{\begin{array}{c} P_{h_i} \\ \hline P_{h_b} \\ \hline P_{h_t} \end{array}\right\}_{(c)} = \left[\begin{array}{c|c|c} a_{h_i k} & 0 & 0 \\ \hline a_{h_b k} & a_{h_b t_p} & b_{h_b t} \\ \hline 0 & a_{h_t t_p} & b_{h_t t} \end{array}\right]_{(c)} \left\{\begin{array}{c} P_k \\ \hline P_{t_p} \\ \hline X_t \end{array}\right\} \qquad \text{(A.33)}$$

in which the [a] matrices are the forces caused by unit applied loads and [b] are forces due to unit connecting redundants. More briefly stated:

$a_{h_i k}$ = Force on h_i point due to unit load at k.

$a_{h_b k}$ = Force on h_b point due to unit load at k.

$a_{h_b t_p}$ = Force on h_b point due to unit load at t_p.

$a_{h_t t_p}$ = Force at h_t point due to unit load on t_p point.

$b_{h_b t}$ = Force on h_b point due to unit connecting redundant at t.

$b_{h_t t}$ = Force on h_t point due to unit connecting redundant at t.

Let the point designation be generalized so that

i = Location of n and s_p points

j = Location of k and t_p points

that is, i and j designate all points where flexibility influence coefficients are calculated so that, for example,

$$\left\{ P_j \right\} = \left\{ \begin{array}{c} P_k \\ \hline P_{t_p} \end{array} \right\} \tag{A.34}$$

Then we can let

$$\left[a_{hj} \right]_{(c)} = \left[\begin{array}{c|c} a_{h_i k} & 0 \\ \hline a_{h_b k} & a_{h_b t_p} \\ \hline 0 & a_{h_t t_p} \end{array} \right]_{(c)} \tag{A.35}$$

and

$$\left[b_{ht}\right]_{(c)} = \begin{bmatrix} b_{h_i t} = [0] \\ \hline b_{h_b t} \\ \hline b_{h_t t} \end{bmatrix} \tag{A.36}$$

Where h now stands for the points h_i, h_b, h_t for which the flexibility matrix γ_{gh} is known from the analysis of each component structures. Then the loads on a particular component structure (c) are:

$$\begin{Bmatrix} P_{h_i} \\ P_{h_b} \\ P_{h_t} \end{Bmatrix}_{(c)} = \left\{P_h\right\}_{(c)} = \left[a_{hj} \;\vdots\; b_{ht}\right]_{(c)} \begin{Bmatrix} P_j \\ \hline X_t \end{Bmatrix} \tag{A.37}$$

2. <u>The Work of Component Structures</u>

The development of a solution for component structures resulted in a method for calculating the component structure flexibility matrix $\left[\gamma_{gh}\right]_{(c)}$ according to Eq. (A.28). The matrix is initially calculated in the desired arrangement or may be suitably rearranged, so that it is partitioned according to the point categories h_i, h_b, h_t previously defined. Actually this is necessary only for clarity of the derivation. Then the deflections relative to the components datum are:

$$\left\{\Delta\right\}_{(c)} = \left[\gamma_{gh}\right]_{(c)} \left\{P_h\right\}_{(c)}$$

or

$$\left\{\begin{array}{c} \Delta_{g_i} \\ \hline \Delta_{g_b} \\ \hline \Delta_{g_t} \end{array}\right\}_{(c)} = \left[\begin{array}{c|c|c} \gamma_{g_i h_i} & \gamma_{g_i h_b} & \gamma_{g_i h_t} \\ \hline \gamma_{g_b h_i} & \gamma_{g_b h_b} & \gamma_{g_b h_t} \\ \hline \gamma_{g_t h_i} & \gamma_{g_t h_b} & \gamma_{g_t h_t} \end{array}\right]_{(c)} \left\{\begin{array}{c} P_{h_i} \\ \hline P_{h_b} \\ \hline P_{h_t} \end{array}\right\}_{(c)} \tag{A.38}$$

The work in the component structure is

$$W_{(c)} = \frac{1}{2}\left[\left\{P_{h_i}\right\}^T\left\{\Delta_{g_i}\right\} + \left\{P_{h_b}\right\}^T\left\{\Delta_{g_b}\right\} + \left\{P_{h_t}\right\}^T\left\{\Delta_{g_t}\right\}\right] \tag{A.39}$$

The work increment due to rigid body motion of the component structure as it displaces due to motion of its datum is zero because the applied forces and the datum reactions are in equilibrium. Therefore, Eq. (A.39) gives one of the components of the total work in the complex structure.

Substituting Eq. (A.37) in Eqs. (A.38) and (A.39), there obtains

$$W_{(c)} = \frac{1}{2}\left[\begin{array}{c|c} P_i & X_s \end{array}\right]\left[\begin{array}{c} a_{ig} \\ \hline b_{sg} \end{array}\right]_{(c)}\left[\gamma_{gh}\right]_{(c)}\left[\begin{array}{c|c} a_{hj} & b_{ht} \end{array}\right]_{(c)}\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.40}$$

in which

$$\left[P_i \mid X_s\right] = \left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\}^T \quad ; \quad \left[\begin{array}{c} a_{ig} \\ \hline b_{sg} \end{array}\right] = \left[a_{hj} \mid b_{ht}\right]^T$$

3. <u>The Work in the Complex Structure</u>

The work in the complex structure is the sum of the works of the components, as given by Eq. (A.40). Thus, for (N) component structures:

$$W_{int} = \sum_{c=1}^{N} W_{(c)}$$

$$W_{int} = \frac{1}{2}\left[P_i \mid X_s\right] \left[\begin{array}{cccc} a_{ig_{(1)}} & a_{ig_{(2)}} & \cdots & a_{ig_{(N)}} \\ \hline b_{sg_{(1)}} & b_{sg_{(2)}} & \cdots & b_{sg_{(N)}} \end{array}\right] \begin{bmatrix} \gamma_{gh_{(1)}} & & \\ & \gamma_{gh_{(2)}} & \\ & & \ddots & \\ & & & \gamma_{gh_{(N)}} \end{bmatrix} \left[\begin{array}{c|c} a_{hj_{(1)}} & b_{ht_{(1)}} \\ a_{hj_{(2)}} & b_{ht_{(2)}} \\ \vdots & \vdots \\ a_{hj_{(N)}} & b_{ht_{(N)}} \end{array}\right] \left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.41}$$

The external work of the complex is calculated as was done for the individual component structure. Assume that a flexibility matrix $[\delta]$ of the complex structure exists. The deflections of the external points i are:

$$\left\{\Delta_i\right\} = \left[\delta_{ij} \mid \delta_{it}\right] \left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.42}$$

Similarly, the relative deflections of the cut points of statically indeterminate interconnection are

$$\left\{\Delta_s\right\} = \left[\delta_{sj} \;\middle|\; \delta_{st}\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.43}$$

With $\left\{\Delta\right\} = \left\{\begin{array}{c} \Delta_i \\ \hline \Delta_s \end{array}\right\}$, it is obvious that

$$\left\{\Delta\right\} = \left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st} \end{array}\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\}$$

Then let the matrix

$$\left[\delta\right] = \left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st} \end{array}\right]$$

so that

$$\left\{\Delta\right\} = \left[\delta\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\}$$

The external work is

$$W_{ext} = \frac{1}{2}\left[P_i \;\middle|\; X_s\right]\left\{\begin{array}{c} \Delta_i \\ \hline \Delta_s \end{array}\right\}$$

so

$$W_{ext} = \frac{1}{2}\left[P_i \;\middle|\; X_s\right]\left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st} \end{array}\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.44}$$

Now $\quad W_{int} = W_{ext}$

Let

$$\left[A_{hj}\right] = \left[\begin{array}{c} a_{hj(1)} \\ a_{hj(2)} \\ \vdots \\ a_{hj(N)} \end{array}\right] \quad \text{and} \quad \left[B_{ht}\right] = \left[\begin{array}{c} b_{ht(1)} \\ b_{ht(2)} \\ \vdots \\ b_{ht(N)} \end{array}\right]$$

Substituting [A] and [B] in Eq. (A.41) and comparing with Eq. (A.44) shows

$$\begin{bmatrix} A_{hj}^{T} \\ \hline B_{ht}^{T} \end{bmatrix} \begin{bmatrix} \diagdown \\ \gamma_{gh} \\ \diagdown \end{bmatrix} \left[A_{hj} \mid B_{ht} \right] = \left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st} \end{array} \right] \tag{A.45}$$

Compatibility requires that the relative deformations $\left\{\Delta_s\right\}$ are equal to zero:

$$\left\{\Delta_s\right\} = \left[\delta_{sj}\right] \left\{P_j\right\} + \left[\delta_{st}\right] \left\{X_t\right\} = 0 \tag{A.46}$$

Therefore

$$\left\{X_t\right\} = -\left[\delta_{st}\right]^{-1} \left[\delta_{sj}\right] \left\{P_j\right\} \tag{A.47}$$

Substituting $\left\{X_t\right\}$ in Eq. (A.42) gives

$$\left\{\Delta_i\right\} = \left[\delta_{ij} \mid \delta_{it}\right] \left\{ \begin{array}{c} P_j \\ \hline -\delta_{st}^{-1} \delta_{sj} P_j \end{array} \right\}$$

or

$$\left\{\Delta_i\right\} = \left[\delta_{ij} - \delta_{it}\, \delta_{st}^{-1}\, \delta_{sj}\right] \left\{P_j\right\} \tag{A.48}$$

Let

$$\left[\gamma_{ij}\right] = \left[\delta_{ij} - \delta_{it}\, \delta_{st}^{-1}\, \delta_{sj}\right] \tag{A.49}$$

Then

$$\left\{\Delta_i\right\} = \left[\gamma_{ij}\right] \left\{P_j\right\} \tag{A.50}$$

which shows that $\left[\gamma_{ij}\right]$ is, by definition, the flexibility influence coefficient matrix of the complex redundant structure.

Let

$$A_{ig} = A_{ht}{}^{T}$$

$$B_{sg} = B_{ht}{}^{T}$$

Then the components of $\left[\gamma_{ij}\right]$ in Eq. (A.49) are obtained from Eq. (A.45) and the substitution of the new terms for A^T, B^T. Thus

$$\delta_{ij} = \left[A_{ig}\right] \left[\diagdown\gamma_{gh}\diagdown\right] \left[A_{hj}\right]$$

$$\delta_{it} = \left[A_{ig}\right] \left[\diagdown\gamma_{gh}\diagdown\right] \left[B_{ht}\right]$$

$$\delta_{sj} = \delta_{it}{}^{T}$$

$$\delta_{st} = \left[B_{sg}\right] \left[\diagdown\gamma_{gh}\diagdown\right] \left[B_{ht}\right]$$

C. SOLUTION SUMMARY

1. Complex Structure

a. Coupling Redundants

$$\left\{X_t\right\} = - \left[\delta_{st}\right]^{-1} \left[\delta_{sj}\right] \left\{P_j\right\} \qquad \text{(A.47)}$$

b. Forces Applied to Component Structures

Substitution of Eq. (A.47) in (A.37)

$$\left\{P_h\right\} = \left[A_{hj} \,\vdots\, B_{ht}\right] \left[\begin{array}{c} I \\ \hline -\delta_{st}{}^{-1}\ \delta_{sj} \end{array}\right] \left\{P_j\right\} \qquad \text{(A.51)}$$

c. Displacements

$$\left\{\Delta_i\right\} = \left[\gamma_{ij}\right] \left\{P_j\right\} \qquad \text{(A.50)}$$

2. Component Structure

Substituting $\left\{P_h\right\}$ of Eq. (A.51) in Eqs. (A.30), (A.31), and (A.32), all internal forces in the component structures can be obtained.

Let

$$x_{qh} = - c_{pq}^{-1} c_{ph} \tag{A.52}$$

(See Equation (A.30))

$$x_{tj} = - \delta_{st}^{-1} \delta_{sj} \tag{A.53}$$

(See Equation (A.47))

$$\begin{aligned} f_{hj} &= \left[a_{hj} - b_{ht} \delta_{st}^{-1} \delta_{sj}\right] \\ &= \left[a_{hj} + b_{ht} x_{tj}\right] \end{aligned} \tag{A.54}$$

(See Equation (A.37), (A.47))

$$f_{mh} = \left[\alpha_{mh} - \beta_{mq} c_{pq}^{-1} c_{ph}\right]$$

or

$$f_{mh} = \left[\alpha_{mh} + \beta_{mq} x_{qh}\right] \tag{A.55}$$

(See Equation (A.31))

Then combining Eqs. (A.52) and (A.54), we obtain component redundants due to unit applied loads as

$$x_{qj} = x_{qh} f_{hj} \tag{A.56}$$

Similarly combining Eqs. (A.54) and (A.55) results in internal forces due to unit applied loads

$$f_{mj} = f_{mh} f_{hj} \tag{A.57}$$

and one obtains the following simplified relationships for the component structure:

Component Structure Redundants

$$\left\{X_q\right\} = \left[x_{qj}\right] \left\{P_j\right\} \tag{A.58}$$

(See Equations (A.30), (A.56))

Component Structure Internal Forces

$$\left\{F_m\right\} = \left[f_{mj}\right] \left\{P_j\right\} \tag{A.59}$$

(See Equations (A.31), (A.57))

Component Structure Deflections

$$\left\{\Delta_g\right\} = \left[\gamma_{gj}\right] \left\{P_j\right\} \tag{A.60}$$

The deflection influence matrix $\left[\gamma_{gj}\right]$ is obtained by choosing those rows of $\left[\gamma_{ij}\right]$ for which $i = g$. The deflections could, of course, have been obtained by choosing the appropriate values from the complex solution, i.e., from $\left\{\Delta_i\right\}$, wherever $i = g$. Formally, the matrix $\left[\gamma_{gj}\right]$ could be obtained by multiplying γ_{gh} by the unit load matrix f_{hj}. From Eq. (A.54),

$$\left[\gamma_{gj}\right] = \left[\gamma_{gh}\right]\left[f_{hj}\right]$$

D. SPECIAL CASES

1. Statically Determinate Coupling of Statically Indeterminate Components

When no coupling redundants X_t exist, $[B]$ does not exist, as can be seen by referring to Eq. (A.33). Following the development of Eq. (A.45) it becomes obvious that only the matrix $\left[\delta_{ij}\right]$ will remain which is given by

$$\left[\delta_{ij}\right] = \left[A_{ig}\right]\left[\diagdown \gamma_{gh} \diagdown\right]\left[A_{hj}\right] \tag{A.61}$$

This matrix is also equal to the flexibility influence matrix $\left[\gamma_{ij}\right]$ of such a system

$$\left[\delta_{ij}\right] = \left[\gamma_{ij}\right]$$

The redundants in each component structure are obtained from Eq. (A.58) in which

$$\left[x_{qj}\right] = \left[x_{qh}\right]\left[f_{hj}\right]$$

$$\left[x_{qh}\right] = - c_{pq}^{-1} c_{ph}$$

$$\left[f_{hj}\right] = \left[a_{hj}\right]$$

so that

$$\left\{X_q\right\} = - \left[c_{pq}\right]^{-1}\left[c_{ph}\right]\left[a_{hj}\right]\left\{P_j\right\} \tag{A.62}$$

The internal forces follow from Eq. (A.59) in which

$$\left[f_{mj}\right] = \left[f_{mh}\right]\left[f_{hj}\right]$$

$$= \left[f_{mh}\right]\left[a_{hj}\right]$$

so that

$$\left\{F_m\right\} = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{-1}\, c_{ph}\right]\left[a_{hj}\right]\left\{P_j\right\} \tag{A.63}$$

2. Statically Determinate Components

If the component structures are statically determinate, their influence coefficient matrices $\left[\gamma_{gh}\right]$ are simply

$$\left[\gamma_{gh}\right]_{S.D.C.} = \left[\alpha_{g\ell}\right]\left[\diagdown\gamma_{\ell m}\right]\left[\alpha_{mh}\right] \tag{A.64}$$

because β_{mq} does not exist. (S.D.C. means "statically determinate component".) For the same reason, $\left[f_{mj}\right] = \left[\alpha_{mh}\right]\left[f_{hj}\right]$, so that the internal forces are:

$$\left\{F_m\right\}_{S.D.C.} = \left[\alpha_{mh}\right]\left[f_{hj}\right]\left\{P_j\right\} \tag{A.65}$$

In case these component structures are coupled statically determinately, $\left[f_{hj}\right] = \left[a_{hj}\right]$, as before, so that in this case

$$\left\{F_m\right\}_{complex} = \left[\alpha_{mh}\right]\left[a_{hj}\right]\left\{P_j\right\} \tag{A.66}$$

3. Summary

The complete solution of a complex structure consisting of statically indeterminate component structures that are attached to each other in a statically indeterminate manner has been found and is summarized in Eqs. (A.47), (A.50), (A.58) and (A.59). Equations (A.47) and (A.58) allow the calculation of the redundants of the system in two stages, first the coupling redundants and subsequently the component structure redundants. Equation (A.50) gives the displacements of the structure and Eq. (A.59) gives the internal forces in the elements of each component structure. The maximum number of redundants at any stage of the computations can be taylored to suit by providing as many or as few component structures as necessary to limit the size of required inversions, as may be seen from Eqs. (A.47) and (A.58). It can also be seen that the matrix $\left[\gamma_{gh}\right]$ of each component structure can be obtained by treating it as a complex structure consisting of sub-components, each of which may or may not be redundant, as the configuration dictates.

Finally, several special relationships were given for the cases in which the coupling of the components is statically determinate, or the components are statically determinate, or both. The method presented has, therefore, great usefulness in those applications in which the complex structure is extremely large and can be clerically handled most easily by assigning a component to each of several groups of personnel. Another advantage is that in a large system that may contain several hundred redundants, it will be possible to break the analysis physically into subdivisions so that no inversions of matrices larger than a size for which good precision can be guaranteed will have to be performed.

Even if the structure is statically determinate, the method presented will allow the calculation of the influence coefficient matrix of very large structures in easy stages, as was shown in the final Eqs. (A.64), (A.65) and (A.66) dealing with this degenerate case.

SECTION IV - SAMPLE PROBLEM

The problem shown in Figure A-13 can be solved in the conventional way by using the theory described in Section III A. This requires the use of three redundants. The alternate coupling procedure is also shown in order to illustrate the use of the theory of Section III B.

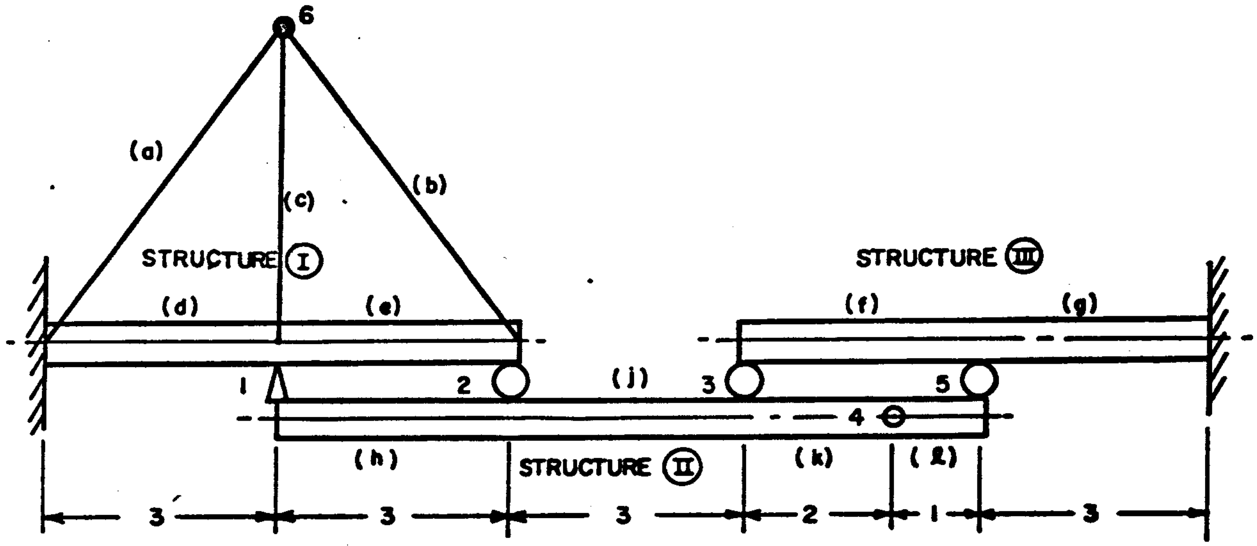


Figure A-13. Problem Idealization

A. CONVENTIONAL MATRIC SOLUTION

Figure A-14 shows two cantilever beams which are fixed in opposite walls, and a third beam which is simply supported at two points on each of these beams. The loads are vertical concentrated forces. The illustration shows the beams in the idealized cut, statically determinate configuration on which this analysis would be based. All beams are assumed to be at the same level.

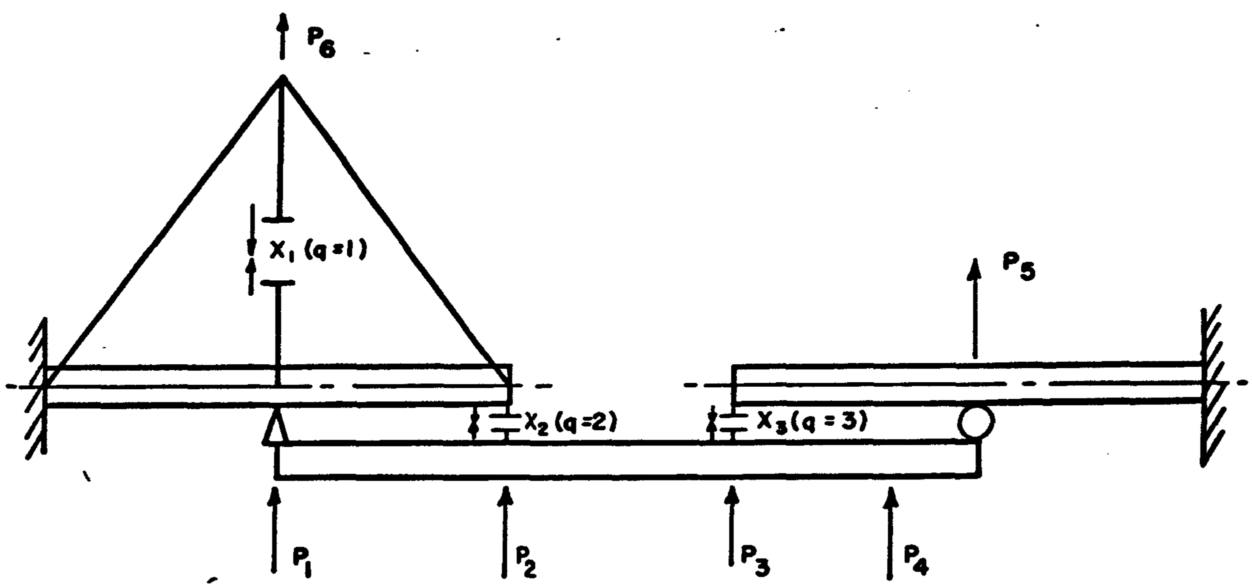


Figure A-14. "Cut" Structure for Conventional Analysis

The free bodies of the structural elements are shown in Figure A-15.

In the conventional analysis the structure is cut at three points, giving a single set of three redundants. The methods of Section III A are applied to obtain a solution.

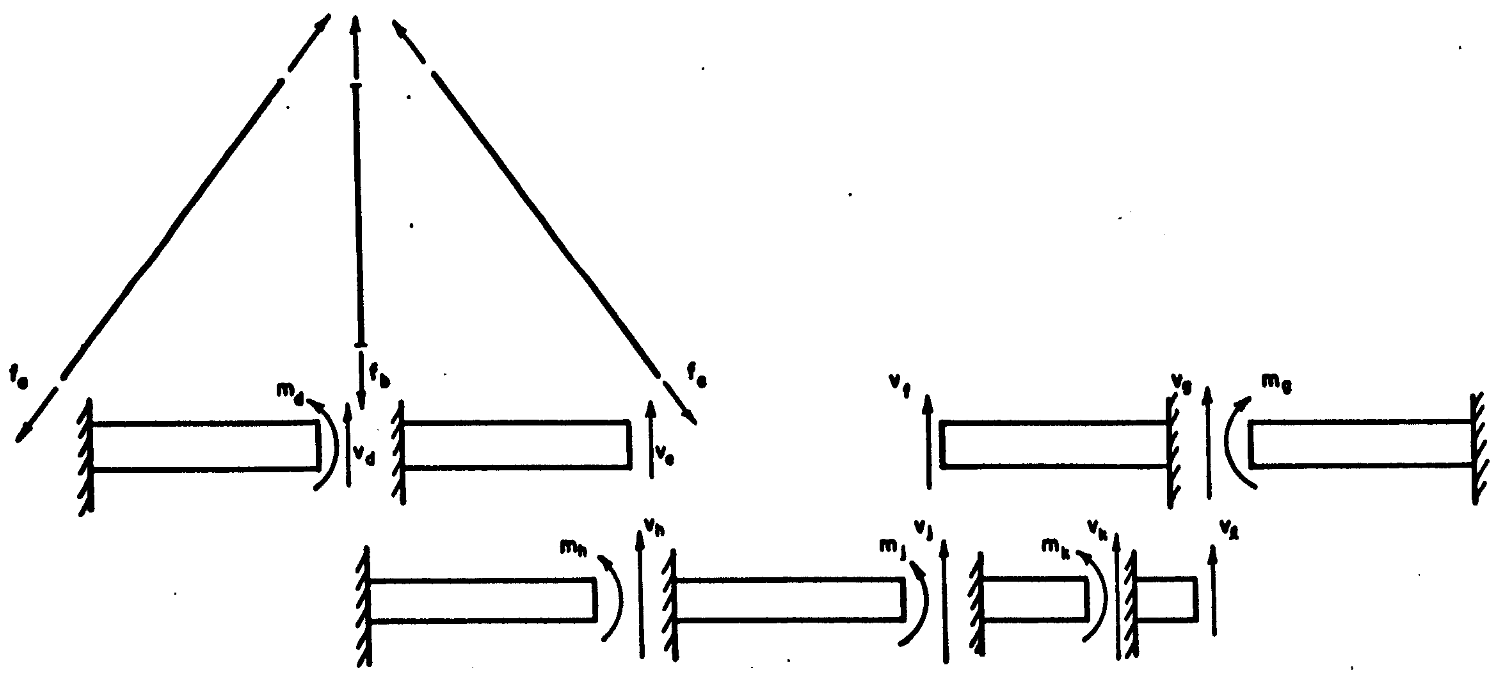


Figure A-15. Free Bodies of Illustrative Problem Elements

1. <u>Internal Force Matrices</u>

		α_{mh}						β_{mq}		
		Load: P_1	P_2	P_3	P_4	P_5	P_6	X_1	X_2	X_3
Load Type	m	h: 1	2	3	4	5	6	q: 1	2	3
f_a	1	0	0	0	0	0	$\frac{5}{8}$	$-\frac{5}{8}$	0	0
f_b	2	0	0	0	0	0	$\frac{5}{8}$	$-\frac{5}{8}$	0	0
f_c	3	0	0	0	0	0	0	1	0	0
v_d	4	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
m_d	5	0	0	0	0	0	$\frac{3}{2}$	$-\frac{3}{2}$	-3	0
v_e	6	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1	0
v_ℓ	7	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{8}{9}$	0	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
v_k	8	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{9}$	0	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
m_k	9	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{8}{9}$	0	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
v_j	10	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	0	0	$-\frac{1}{3}$	$\frac{1}{3}$
m_j	11	0	-1	-2	$-\frac{2}{3}$	0	0	0	-1	-2
v_h	12	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	0	0	$\frac{2}{3}$	$\frac{1}{3}$
m_h	13	0	-2	-1	$-\frac{1}{3}$	0	0	0	-2	-1
v_f	14	0	0	0	0	0	0	0	0	-1
v_g	15	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{8}{9}$	1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$
m_g	16	0	0	0	0	0	0	0	0	-3

2. <u>Element Flexibility Matrices</u>

Element (a): $L = 5$ $EA = 1$

$$\gamma = \frac{L}{EA} = 5$$

(b) $L = 5$ $EA = 1$

$$\gamma = \frac{L}{EA} = 5$$

(c) $L = 4$ $EA = 1$

$$\gamma = \frac{L}{EA} = 4$$

$\left.\begin{matrix}\text{(d), (e), (h)}\\ \text{(j), (f), (g)}\end{matrix}\right\}$ $L = 3$ $EI = 1$

$$\gamma_{11} = \frac{L^3}{3EI} = 9 \qquad \gamma_{12} = \gamma_{21} = \frac{L^2}{2EI} = 4.5$$

$$\gamma_{22} = \frac{L}{EI} = 3$$

(k) $L = 2$ $EI = 1$

$$\gamma_{11} = \frac{L^3}{3EI} = \frac{8}{3} \qquad \gamma_{12} = \gamma_{21} = \frac{L^2}{2EI} = 2$$

$$\gamma_{22} = \frac{L}{EI} = 2$$

(l) $L = 1$ $EI = 1$

$$\frac{L^3}{3EI} = \frac{1}{3}$$

$$
[\gamma_{\ell m}] = \begin{array}{cc}
{}_{\ell}\backslash^{m} & \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array} &
\left[\begin{array}{cccccccccccccccc}
5 & & & & & & & & & & & & & & & \\
 & 5 & & & & & & & & & & & & & & \\
 & & 4 & & & & & & & & & & & & & \\
 & & & 9 & 4.5 & & & & & & & & & & & \\
 & & & 4.5 & 3 & & & & & & & & & & & \\
 & & & & & 9 & & & & & & & & & & \\
 & & & & & & \frac{1}{3} & & & & & & & & & \\
 & & & & & & & \frac{8}{3} & 2 & & & & & & & \\
 & & & & & & & 2 & 2 & & & & & & & \\
 & & & & & & & & & 9 & 4.5 & & & & & \\
 & & & & & & & & & 4.5 & 3 & & & & & \\
 & & & & & & & & & & & 9 & 4.5 & & & \\
 & & & & & & & & & & & 4.5 & 3 & & & \\
 & & & & & & & & & & & & & 9 & & \\
 & & & & & & & & & & & & & & 9 & 4.5 \\
 & & & & & & & & & & & & & & 4.5 & 3
\end{array}\right]
\end{array}
$$

B. SOLUTION THROUGH COUPLING OF COMPONENT STRUCTURES

This solution is obtained by applying the methods of Section III B. Structure I is a once-redundant component structure. The assemblage of the three component structures has statically determinate interconnection when cuts are made at points 2 and 3. The coupling matrices [A] and [B] for the cut structure of Figure A-16 are shown on page 51.

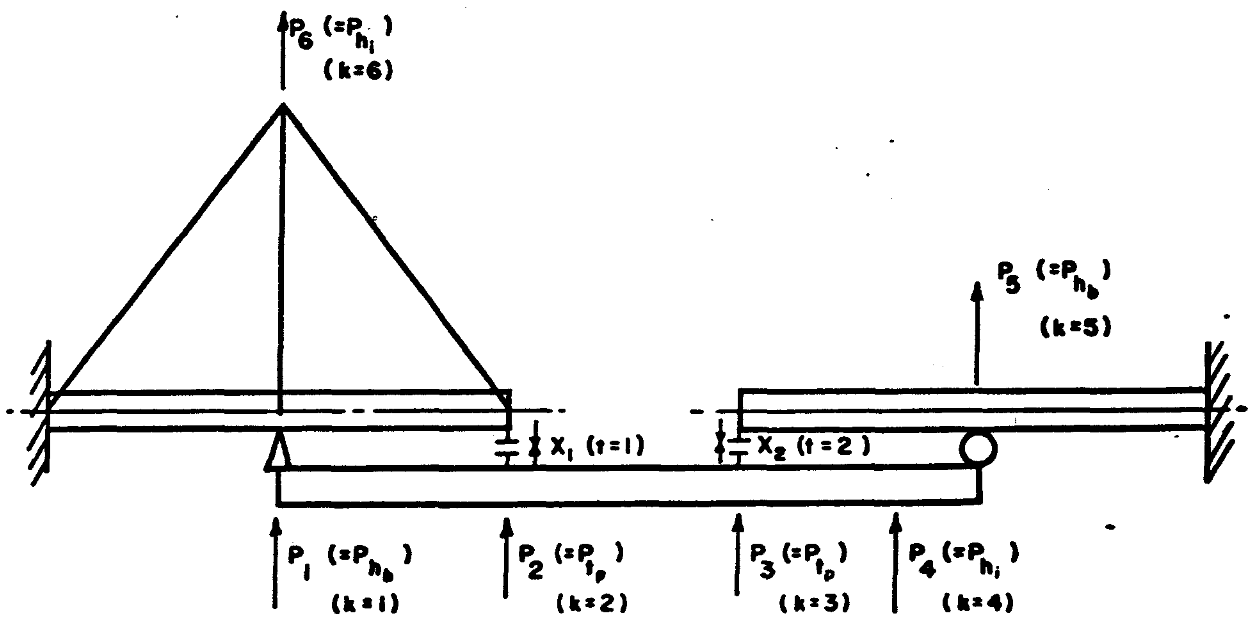
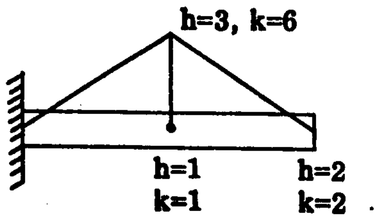
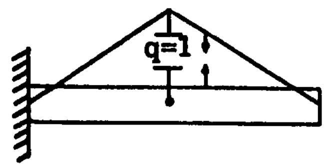


Figure A-16. Illustrative Problem

The component internal force matrices are given first:

Component I

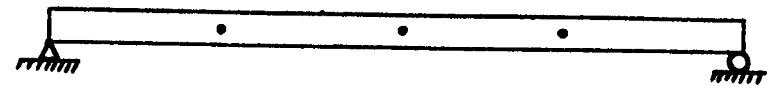




$[\alpha_{mh}]_I$ $[\beta_{mq}]_I$

Force Type	m	h = 1	h = 2	h = 3	q = 1	q = (X_3)
f_a	1	0	0	$\frac{5}{8}$	$-\frac{5}{8}$	
f_b	2	0	0	$\frac{5}{8}$	$-\frac{5}{8}$	
f_c	3	0	0	0	1	
v_d	4	1	1	$\frac{1}{2}$	$\frac{1}{2}$	
m_d	5	0	3	$\frac{3}{2}$	$-\frac{3}{2}$	
v_e	6	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	

Component II



h = 4 k = 1 h = 1 k = 2 h = 2 k = 3 h = 3 k = 4 h = 5 k = 5

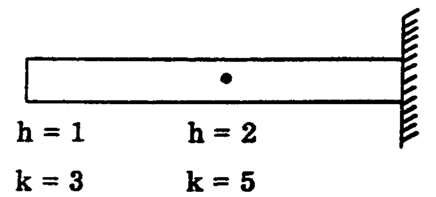
$[\alpha_{mh}]_{II}$

Force Type	m	h = 1	h = 2	h = 3
v_ℓ	1	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{8}{9}$
v_k	2	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{9}$
m_k	3	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{8}{9}$
v_j	4	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
m_j	5	-1	-2	$-\frac{2}{3}$
v_h	6	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
m_h	7	-2	-1	$-\frac{1}{3}$

Component III

$$\left[\alpha_{mh}\right]_{III}$$

Force Type	m	h = 1	h = 2
v_f	1	1	0
v_g	2	1	1
m_g	3	3	0



The Coupling Matrices [A] and [B]

(Refer to Fig. A-16) $\left[A_{hj}\right]$ $\left[B_{ht}\right]$

Component No.	Point Type	j		k						t	
				1	2	3	4	5	6	1	2
			Force Type →	j′	t_p	t_p	j′	j′	j′	X_1	X_2
I	h_b	1		1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$
	h_t	2		0	0	0	0	0	0	-1	0
	h_j	6		0	0	0	0	0	1	0	0
II	h_t	2		0	1	0	0	0	0	1	0
	h_t	3		0	0	1	0	0	0	0	1
	h_j	4		0	0	0	1	0	0	0	0
III	h_t	3		0	0	0	0	0	0	0	-1
	h_b	5		0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{8}{9}$	1	0	$\frac{1}{3}$	$\frac{2}{3}$

The Component Element Flexibility Matrices

These matrices correspond to the diagonal partitions of $\gamma_{\ell m}$, with partitioning between 6 and 7 and between 13 and 14, because the arrangement of ℓ and m indices corresponds to that of the components. Thus

$$\left[\gamma_{\ell m}\right]_{I} = \begin{bmatrix} 5 & & & & & \\ & 5 & & & & \\ & & 4 & & & \\ & & & 9 & 4.5 & \\ & & & 4.5 & 3 & \\ & & & & & 9 \end{bmatrix}$$

$$\left[\gamma_{\ell m}\right]_{II} = \begin{bmatrix} \frac{1}{3} & & & & & & \\ & \frac{8}{3} & 2 & & & & \\ & 2 & 2 & & & & \\ & & & 9 & 4.5 & & \\ & & & 4.5 & 3 & & \\ & & & & & 9 & 4.5 \\ & & & & & 4.5 & 3 \end{bmatrix}$$

$$\left[\gamma_{\ell m}\right]_{III} = \begin{bmatrix} 9 & & \\ & 9 & 4.5 \\ & 4.5 & 3 \end{bmatrix}$$

APPENDIX A - PROOF OF EQUATION (A.25)

To prove:

$$\left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = \left[\begin{array}{c} \alpha_{mh}{}^{T} \\ \hline \beta_{mq}{}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\alpha_{mh} \mid \beta_{mq}\right]$$

From Eq. (A.20) and (A.24),

$$\left[P_h{}^T \mid X_q{}^T\right] \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} = \left[P_h{}^T \mid X_q{}^T\right] \left[\begin{array}{c} \alpha_{mh}{}^{T} \\ \hline \beta_{mq}{}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\alpha_{mh} \mid \beta_{mq}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

Let

$$\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} = \{L\}, \quad \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = [c], \quad \left[\alpha_{mh} \mid \beta_{mq}\right] = [d],$$

$$\left[\diagdown \gamma_{\ell m} \diagdown\right] = \gamma$$

Then

$$\left[L^T\right] \left[c\right] \left[L\right] = \left[L^T\right] \left[d^T \gamma d\right] \left[L\right]$$

Subtracting equal quantities from both sides:

$$\left[L^T\right] \left[c - d^T \gamma d\right] \left[L\right] = 0$$

but $\quad L \neq 0$

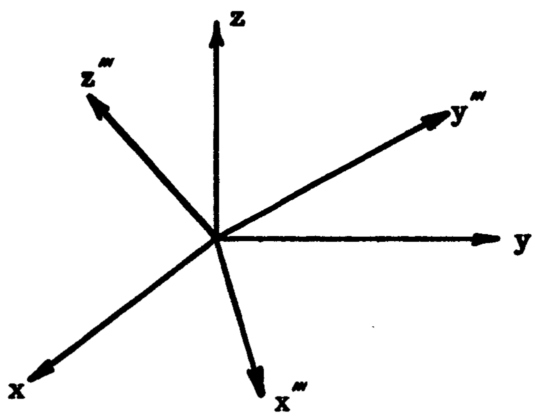
therefore $\quad c = d^T \gamma d$

or

$$\left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = \left[\begin{array}{c} \alpha_{mh}{}^{T} \\ \hline \beta_{mq}{}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\alpha_{mh} \mid \beta_{mq}\right]$$

APPENDIX B - AXIS SYSTEM TRANSFORMATIONS OF FORCE AND DEFORMATION VECTORS

Suppose a structural element to exist with orthogonal principal axes x''', y''', z''', oriented arbitrarily with respect to a common Cartesian coordinate system as shown in the following sketch.



Arbitrary Location of Triple Prime Axis System with Respect to Common x, y, z System

Let it be so oriented that the direction cosines of the x''' axis are given by ℓ_x, m_x, and n_x, respectively, being the cosines of the angles between the x''' axis and the x, y, and z axes. Similarly ℓ_y, m_y, and n_y are the direction cosines of the angles between the y''' axis and the x, y, and z axes, respectively. The subscripts z refer to similar quantities pertaining to location of the z axis. Thus it can easily be seen that the transformation of forces from one axis system, x''', y''', z''' to the common Cartesian system is obtained by the matrix multiplication

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{bmatrix} \ell_x & \ell_y & \ell_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{Bmatrix} F_x''' \\ F_y''' \\ F_z''' \end{Bmatrix}$$

Forces and deformations which are alike in sense and direction transform in the same way, through the previously shown direction cosine matrix, in which we let

$$[T] = \begin{bmatrix} \ell_x & \ell_y & \ell_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}$$

Methods for obtaining the direction cosines are straightforward and follow from the calculation of unit vectors along chosen body axes. It can be shown that this transformation matrix is orthonormal, so that its inverse is equal to its transpose.

$$[T]^{-1} = [T]^T$$

In order to ensure the orthogonality of the T vectors, the following orthogonality conditions must be met:

Let $v_i = \begin{Bmatrix} \ell_i \\ m_i \\ n_i \end{Bmatrix}$; $i = x, y, z$

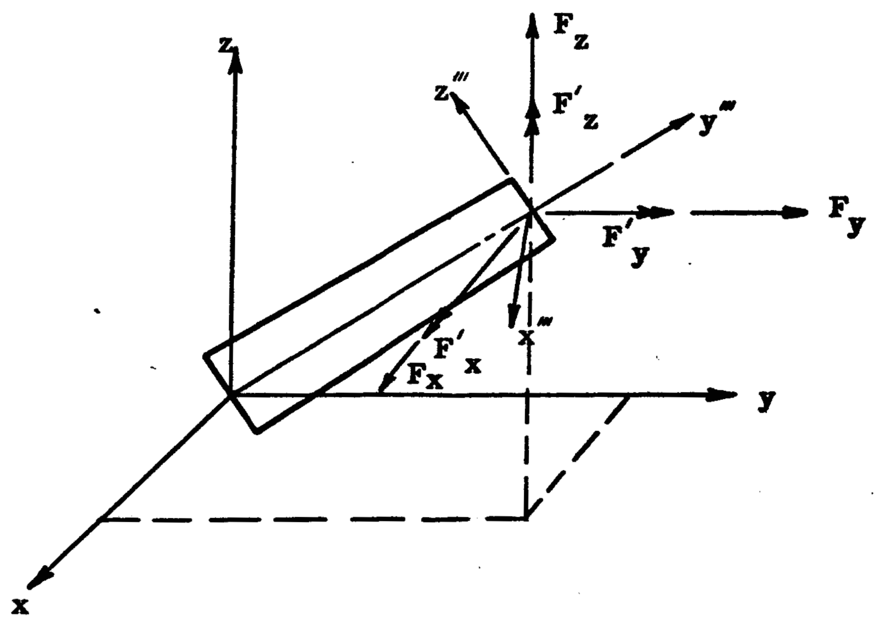
Then $\sum\limits_n v_{ni} v_{nj} = \delta_{ij}$

where $\delta_{ij} = 1;\ i = j$
$\quad\quad = 0;\ i \neq j$

APPENDIX C - EXTERNAL TO LOCAL INTERNAL FORCE TRANSFORMATION

1. Transformation from Common into Local Element Axis System

In the calculation of the influence coefficient matrix of a complex structure it will be necessary to transform forces given in the common x, y, z coordinate system into those acting along the principal axes of the element. Suppose six forces are given at the end of the element shown in the following sketch.



Prismatic Arbitrarily Located Element Subjected to End Forces

Knowing

$$\begin{Bmatrix} F_{x'''} \\ F_{y'''} \\ F_{z'''} \end{Bmatrix} = \left[T \right]^T \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

we arrange the forces which are in the body axis system into two triplets of forces and moments. The resulting transformation yields the formula:

$$\begin{Bmatrix} F'_{x'''} \\ F'_{y'''} \\ F'_{z'''} \\ \hline F_{x'''} \\ F_{y'''} \\ F_{z'''} \end{Bmatrix} = \left[\begin{array}{c|c} [T]^T & [0] \\ \hline [0] & [T]^T \end{array}\right] \begin{Bmatrix} F'_x \\ F'_y \\ F'_z \\ \hline F_x \\ F_y \\ F_z \end{Bmatrix}$$

where $[T]^T$ is obtained as previously discussed.

$F'_{x'''}$ = moment about x''' axis

$F'_{y'''}$ = torque about y''' axis

$F'_{z'''}$ = moment about z''' axis

$F_{x'''}$ = transverse shear in x''' axis direction

$F_{y'''}$ = axial force in y''' axis direction

$F_{z'''}$ = shear force in z''' axis direction

2. Transformation of Externally Applied Loads into Internal Element Forces

The forces at the ends of structural elements that are caused by externally applied loads anywhere on the complex structure must be calculated so that the internal work in the elements can be calculated. The use of internal work in the application of the principle of virtual work for the solution of complex redundant structures was shown in Section III. The internal force influence matrices occur in two types. One is the matrix whose elements are the element forces due to unit values of the applied loads; the other type gives the element forces due to unit values of the redundants.

The determination of these matrices obviously requires that the structure be cut until a stable, statically determinate structure remains. Then the internal forces can be obtained from the equations of static equilibrium, which involve only the geometry.

The cutting should be done so that, if possible, the redundants will have the least effect on the internal forces in the structure.

Let $\left[\alpha\right]$ = Influence matrix giving internal forces due to unit values of applied load

$\left[\beta\right]$ = Influence matrix giving internal forces due to unit values of redundant forces acting simultaneously on both sides of one cut

Then the internal forces are:

$$\left\{\begin{array}{c} F_{g'} \\ \hline F_g \end{array}\right\} = \left[\begin{array}{c|c} \alpha_{g'h} & \beta_{g'q} \\ \hline \alpha_{gh} & \beta_{gq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

where P_h = external forces at h

X_q = redundant forces at q

But

$$\left\{F_{i''}\right\} = \left[T\right]^T \left\{F_i\right\}.$$

Thus

$$\left\{\begin{array}{c} F'_{g''} \\ \hline F_{g''} \end{array}\right\} = \left[\begin{array}{c|c} \left[T\right]^T & \left[0\right] \\ \hline \left[0\right] & \left[T\right]^T \end{array}\right] \left[\begin{array}{c|c} \alpha_{g'h} & \beta_{g'q} \\ \hline \alpha_{gh} & \beta_{gq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

These are the desired internal forces in the principal axis system of the structural element. As an example, consider the force transformation for a rod element

$$\left\{ F_{g''} \right\} = \left[T \right]^T_{rod} \left[\alpha_{gh} \;\vdots\; \beta_{gq} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\}$$

where $\left\{ F_{g''} \right\} = \left\{ F_{y''} \right\}$

$$\left[T \right]^T_{rod} = \left[\ell \quad m \quad n \right]$$

$$\left\{ \begin{array}{c} F_x \\ F_y \\ F_z \end{array} \right\} = \left[\alpha_{gh} \;\vdots\; \beta_{gq} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\}$$

and ℓ, m, n are the direction cosines of the rod axis (y'') with respect to the x, y, z system.

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